The Mean Value Theorem for Divergence Form Elliptic Operators

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February 27, 2013

Abstract

In 1963, Littman, Stampacchia, and Weinberger proved a mean value theorem for elliptic operators in divergence form with bounded measurable coefficients. In the Fermi lectures in 1998, Caffarelli stated a much simpler mean value theorem for the same situation, but did not include the details of the proof. We show all of the nontrivial details needed to prove the formula stated by Caffarelli.

1 Introduction

Based on the ubiquitous nature of the mean value theorem in problems involving the Laplacian, it is clear that an analogous formula for a general divergence form elliptic operator would necessarily be very useful. In [LSW], Littman, Stampacchia, and Weinberger stated a mean value theorem for a general divergence form operator, L. If μ is a nonnegative measure on Ω and u is the solution to:

$$\begin{aligned} Lu = \mu \quad & \text{in} \;\; \Omega \\ 0 \quad & \text{on} \;\; \partial\Omega \; , \end{aligned} \tag{1.1}$$

and $G(x, y)$ is the Green's function for L on Ω then Equation 8.3 in their paper states that $u(y)$ is equal to

$$\lim_{a\to\infty} \frac{1}{2a} \int_{a \leq G \leq 3a} u(x) a^{ij}(x) D_{x_i} G(x, y) D_{x_j} G(x, y) \; dx \tag{1.2}$$

almost everywhere, and this limit is nondecreasing. The pointwise definition of u given by this equation is necessarily lower semi-continuous. There are a few reasons why this formula is not as nice as the basic mean value formulas for Laplace's equation. First, it is a weighted average and not a simple average. Second, it is not an average over a ball or something which is even

homeomorphic to a ball. Third, it requires knowledge of derivatives of the Green's function.

A simpler formula was stated by Caffarelli in [C]. That formula provides an increasing family of sets, $D_R(x_0)$, which are each comparable to B_R and such that for a supersolution to $Lu = 0$ the average:

$$\frac{1}{|D_R(x_0)|}\int_{D_R(x_0)} u(x)\,dx$$

is nondecreasing as $R \to 0$. On the other hand, Caffarelli did not provide any details about showing the existence of an important test function used in the proof of this result, and showing the existence of this function turns out to be nontrivial. This paper grew out of an effort to prove rigorously all of the details of the mean value theorem that Caffarelli asserted in [C].

2 Assumptions

We assume that $a^{ij}(x)$ satisfy

$$a^{ij} \equiv a^{ji} \;\text{ and }\; 0 < \lambda|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \;\text{ for all } \xi \in \mathbb{R}^n,\; \xi \neq 0\;,$$

and to ease the exposition, we assume that $n \geq 3$. Throughout the entire paper, n, λ, and Λ will remain fixed, and so we will omit all dependence on these constants in the statements of our theorems. We define the divergence form elliptic operator

$$L := D_j\, a^{ij}(x)D_i\;, \tag{2.1}$$

or, in other words, for a function $u \in W^{1,2}(\Omega)$ and $f \in L^2(\Omega)$ we say "$Lu = f$ in Ω" if for any $\phi \in W_0^{1,2}(\Omega)$ we have:

$$-\int_\Omega a^{ij}(x)D_iuD_j\phi = \int_\Omega g\phi\;. \tag{2.2}$$

(Notice that with our sign conventions we can have $L = \Delta$ but not $L = -\Delta$.) With our operator L we let $G(x,y)$ denote the Green's function for all of $\mathbb{R}^n$ and observe that the existence of G is guaranteed by the work of Littman, Stampacchia, and Weinberger. (See [LSW].)
Let

$$C_{sm,r} := \min_{x\in\partial B_r} G(x,0)$$
$$C_{big,r} := \max_{x\in\partial B_r} G(x,0)$$
$$G_{sm,r}(x) := \min\{G(x,0), C_{sm,r}\}$$

and observe that $G_{sm,r} \in W^{1,2}(B_M)$ by results from [LSW] combined with the Caccioploi Energy Estimate. We also know that there is an $\alpha \in (0,1)$ such that $G_{sm,r} \in C^{0,\alpha}(\overline{B_M})$ by the De Giorgi-Nash-Moser theorem. (See [GT] or [HL] for example.) For M large enough to guarantee that $G_{sm}(x) := G_{sm,1}(x) \equiv G(x,0)$ on ∂B_M, we define:

$$H_{M,G} := \{ w \in W^{1,2}(B_M) \ : \ w - G_{sm} \in W_0^{1,2}(B_M) \ \}$$

and

$$K_{M,G} := \{ \ w \in H_{M,G} \ : \ w(x) \leq G(x,0) \text{ for all } x \in B_M \ \}.$$

(The existence of such an M follows from [LSW], and henceforth any constant M will be large enough so that $G_{sm,1}(x) \equiv G(x,0)$ on ∂B_M.)

Define:

$$\Phi_\epsilon(t) := \begin{cases} 0 & \text{for } t \geq 0 \\ -\epsilon^{-1} t & \text{for } t \leq 0 \ , \end{cases}$$

$$J(w,\Omega) := \int_\Omega (a^{ij} D_i w D_j w - 2R^{-n} w) \ , \quad \text{and}$$

$$J_\epsilon(w,\Omega) := \int_\Omega (a^{ij} D_i w D_j w - 2R^{-n} w + 2\Phi_\epsilon(G - w)) \ .$$

3 The PDE Satisfied by Minimizers

There are two main points to this section. First, we deal with the comparatively simple task of getting existence, uniqueness, and continuity of certain minimizers to our functionals in the relevent sets. Second, and more importantly we show that the minimizer is the solution of an obstacle type free boundary problem.

3.1 Theorem (Existence and Uniqueness)**.**

$$\textit{Let } \ell_0 := \inf_{w \in K_{M,G}} J(w, B_M) \quad \textit{and}$$

$$\textit{let } \ell_\epsilon := \inf_{w \in H_{M,G}} J_\epsilon(w, B_M) \ .$$

Then there exists a unique $w_0 \in K_{M,G}$ such that $J(w_0, B_M) = \ell_0$, and there exists a unique $w_\epsilon \in H_{M,G}$ such that $J_\epsilon(w_\epsilon, B_M) = \ell_\epsilon$.

Proof. Both of these results follow by a straightforward application of the direct method of the Calculus of Variations. ∎

3.2 Remark. Notice that we cannot simply minimize either of our functionals on all of $\mathbb{R}^n$ instead of B_M as the Green's function is not integrable at infinity. Indeed, if we replace B_M with $\mathbb{R}^n$ then

$$\ell_0 = \ell_\epsilon = -\infty$$

and so there are many technical problems.

3.3 Theorem (Continuity)**.** *For any $\epsilon > 0$, the function w_ϵ is continuous on $\overline{B_M}$.*

See Chapter 7 of [G].

3.4 Lemma. *There exists $\epsilon > 0$, $C < \infty$, such that $w_0 \leq C$ in B_ϵ.*

Proof. Let $\bar{w}$ minimize $J(w, B_M)$ among functions $w \in H_{M,G}$. Then we have

$$w_0 \leq \bar{w}.$$

Set $b := C_{big,M} = \max_{\partial B_M} G(x, 0)$, and let w_b minimize $J(w, B_M)$ among $w \in W^{1,2}(B_M)$ with

$$w - b \in W_0^{1,2}(B_M).$$

Then by the weak maximum principle, we have

$$\bar{w} \leq w_b.$$

Next define $\ell(x)$ by

$$\ell(x) := b + R^{-n}\left(\frac{M^2 - |x|^2}{4n}\right) \leq b + \frac{R^{-n}M^2}{4n} < \infty. \tag{3.1}$$

With this definition, we can observe that ℓ satisfies

$$\Delta\ell = -\frac{R^{-n}}{2}, \text{ in } B_M \quad \text{and}$$
$$\ell \equiv b := \max_{\partial B_M} G \text{ on } \partial B_M.$$

Now let $\widetilde{\alpha}$ be $b + \frac{R^{-n}M^2}{4n}$. By Corollary 7.1 in [LSW] applied to $w_b - b$ and $\ell - b$, we have

$$w_b \leq b + K(\ell - b) \leq b + K\widetilde{\alpha} < \infty.$$

Chaining everything together gives us

$$w_0 \leq b + K\widetilde{\alpha} < \infty.$$

■

3.5 Lemma. *If* $0 < \epsilon_1 \leq \epsilon_2$, *then*

$$w_{\epsilon_1} \leq w_{\epsilon_2}.$$

Proof. Assume $0 < \epsilon_1 \leq \epsilon_2$, and assume that

$$\Omega_1 := \{w_{\epsilon_1} > w_{\epsilon_2}\}$$

is not empty. Since $w_{\epsilon_1} = w_{\epsilon_2}$ on ∂B_M, since $\Omega_1 \subset B_M$, and since w_{ϵ_1} and w_{ϵ_2} are continuous functions, we know that $w_{\epsilon_1} = w_{\epsilon_2}$ on $\partial\Omega_1$. Then it is clear that among functions with the same data on $\partial\Omega_1$, w_{ϵ_1} and w_{ϵ_2} are minimizers of $J_{\epsilon_1}(\cdot, \Omega_1)$ and $J_{\epsilon_2}(\cdot, \Omega_1)$ respectively. Since we will restrict our attention to Ω_1 for the rest of this proof, we will use $J_\epsilon(w)$ to denote $J_\epsilon(w, \Omega_1)$.

$J_{\epsilon_2}(w_{\epsilon_2}) \leq J_{\epsilon_2}(w_{\epsilon_1})$ implies

$$\begin{aligned}
&\int_{\Omega_1} a^{ij} D_i w_{\epsilon_2} D_j w_{\epsilon_2} - 2R^{-n} w_{\epsilon_2} + 2\Phi_{\epsilon_2}(G - w_{\epsilon_2}) \\
\leq &\int_{\Omega_1} a^{ij} D_i w_{\epsilon_1} D_j w_{\epsilon_1} - 2R^{-n} w_{\epsilon_1} + 2\Phi_{\epsilon_2}(G - w_{\epsilon_1}) \ ,
\end{aligned}$$

and by rearranging this inequality we get

$$\int_{\Omega_1} (a^{ij} D_i w_{\epsilon_2} D_j w_{\epsilon_2} - 2R^{-n} w_{\epsilon_2}) - \int_{\Omega_1} (a^{ij} D_i w_{\epsilon_1} D_j w_{\epsilon_1} - 2R^{-n} w_{\epsilon_1})$$

$$\leq \int_{\Omega_1} 2\Phi_{\epsilon_2}(G - w_{\epsilon_1}) - 2\Phi_{\epsilon_2}(G - w_{\epsilon_2}) \ .$$

Therefore,

$$\begin{aligned}
&J_{\epsilon_1}(w_{\epsilon_2}) - J_{\epsilon_1}(w_{\epsilon_1}) \\
= &\int_{\Omega_1} a^{ij} D_i w_{\epsilon_2} D_j w_{\epsilon_2} - 2R^{-n} w_{\epsilon_2} + 2\Phi_{\epsilon_1}(G - w_{\epsilon_2}) \\
&\quad - \int_{\Omega_1} a^{ij} D_i w_{\epsilon_1} D_j w_{\epsilon_1} - 2R^{-n} w_{\epsilon_1} + 2\Phi_{\epsilon_1}(G - w_{\epsilon_1}) \\
\leq &\ 2\int_{\Omega_1} \left[\Phi_{\epsilon_2}(G - w_{\epsilon_1}) - \Phi_{\epsilon_2}(G - w_{\epsilon_2})\right] \\
&\quad - 2\int_{\Omega_1} \left[\Phi_{\epsilon_1}(G - w_{\epsilon_1}) - \Phi_{\epsilon_1}(G - w_{\epsilon_2})\right] \\
< &\ 0
\end{aligned}$$

since $G - w_{\epsilon_1} < G - w_{\epsilon_2}$ in Ω_1 and Φ_{ϵ_1} decreases as fast or faster than Φ_{ϵ_2} decreases everywhere. This inequality contradicts the fact that w_{ϵ_1} is the minimizer of $J_{\epsilon_1}(w)$. Therefore, $w_{\epsilon_1} \leq w_{\epsilon_2}$ everywhere in Ω. ■

3.6 Lemma. *$w_0 \leq w_\epsilon$ for every $\epsilon > 0$.*

Proof. Let $S := \{w_0 > w_\epsilon\}$ be a nonempty set, let $w_1 := \min\{w_0, w_\epsilon\}$, and let $w_2 := \max\{w_0, w_\epsilon\}$. It follows that $w_1 \leq G$ and both w_1 and w_2 belong to $W^{1,2}(B_M)$. Since $\Phi_\epsilon \geq 0$, we know that for any $\Omega \subset B_M$ we have

$$J(w, \Omega) \leq J_\epsilon(w, \Omega) \tag{3.2}$$

for any permissible w. We also know that since $w_0 \leq G$ we have:

$$J(w_0, \Omega) = J_\epsilon(w_0, \Omega)\ . \tag{3.3}$$

Now we estimate:

$$\begin{aligned}
J_\epsilon(w_1, B_M) &= J_\epsilon(w_1, S) + J_\epsilon(w_1, S^c) \\
&= J_\epsilon(w_\epsilon, S) + J_\epsilon(w_0, S^c) \\
&= J_\epsilon(w_\epsilon, B_M) - J_\epsilon(w_\epsilon, S^c) + J_\epsilon(w_0, S^c) \\
&\leq J_\epsilon(w_2, B_M) - J_\epsilon(w_\epsilon, S^c) + J_\epsilon(w_0, S^c) \\
&= J_\epsilon(w_0, S) + J_\epsilon(w_\epsilon, S^c) - J_\epsilon(w_\epsilon, S^c) + J_\epsilon(w_0, S^c) \\
&= J_\epsilon(w_0, S) + J_\epsilon(w_0, S^c) \\
&= J_\epsilon(w_0, B_M)\ .
\end{aligned}$$

Now by combining this inequality with Equations (3.2) and (3.3), we get:

$$J(w_1, B_M) \leq J_\epsilon(w_1, B_M) \leq J_\epsilon(w_0, B_M) = J(w_0, B_M)\ ,$$

but if S is nonempty, then this inequality contradicts the fact that w_0 is the unique minimizer of J among functions in $K_{M,G}$. ■

Now, since w_ϵ decreases as $\epsilon \to 0$, and since the w_ϵ's are bounded from below by w_0, there exists

$$\widetilde{w} = \lim_{\epsilon \to 0} w_\epsilon$$

and $w_0 \leq \widetilde{w}$.

3.7 Lemma. *With the definitions as above, $\tilde{w} \leq G$ almost everywhere.*

Proof. This fact is fairly obvious, and the proof is fairly straightforward, so we supply only a sketch.

Suppose not. Then there exists an $\alpha > 0$ such that

$$\tilde{S} := \{\tilde{w} - G \geq \alpha\}$$

has positive measure. On this set we automatically have $w_\epsilon - G \geq \alpha$. We compute $J_\epsilon(w_\epsilon, B_M)$ and send ϵ to zero. We will get $J_\epsilon(w_\epsilon, B_M) \to \infty$ which gives us a contradiction. ■

3.8 Lemma. $\widetilde{w} = w_0$ *in* $W^{1,2}(B_M)$.

Proof. Since for any ϵ, w_ϵ is the minimizer of $J_\epsilon(w, B_M)$, we have

$$\begin{aligned} J_\epsilon(w_\epsilon, B_M) &\leq J_\epsilon(w_0, B_M) \\ &\leq \int_{B_M} a^{ij} D_i w_0 D_j w_0 - 2R^{-n} w_0 + 2\Phi_\epsilon(G - w_\epsilon), \end{aligned}$$

and after canceling the terms with Φ_ϵ we have:

$$\int_{B_M} a^{ij} D_i w_\epsilon D_j w_\epsilon - 2R^{-n} w_\epsilon \leq \int_{B_M} a^{ij} D_i w_0 D_j w_0 - 2R^{-n} w_0.$$

Letting $\epsilon \to 0$ gives us

$$J(\tilde{w}, B_M) \leq J(w_0, B_M)\ .$$

However, by Proposition (3.7), $\widetilde{w}$ is a permissible competitor for the problem $\inf_{w \in K_{M,G}} J(w, B_M)$, so we have

$$J(w_0, B_M) \leq J(\tilde{w}, B_M).$$

Therefore

$$J(w_0, B_M) = J(\tilde{w}, B_M),$$

and then by uniqueness, $\widetilde{w} = w_0$. ■

Let W solve:

$$\begin{cases} L(w) = -\chi_{\{w<G\}} R^{-n} & \text{in } \ B_M \\ \\ w = G_{sm} & \text{on } \ \partial B_M\ . \end{cases} \tag{3.4}$$

3.9 Lemma. $W \leq G$ *in* B_M.

Proof. Let $\Omega = \{W > G\}$ and $u := W - G$. Since G is infinite at 0, and since W is bounded, and both G and W are continuous, we know there exists an $\epsilon > 0$ such that $\Omega \cap B_\epsilon = \phi$. Then if $\Omega \neq \phi$, then u has a positive maximum in the interior of Ω. However, since $L(W) = L(G) = 0$ in Ω, we would get a contradiction from the weak maximum principle. Therefore, we have $W \leq G$ in B_M. ■

3.10 Lemma. $\tilde{w} \geq W$.

Proof. It suffices to show $w_\epsilon \geq W$, for any ϵ. Suppose for the sake of obtaining a contradiction that there exists an $\epsilon > 0$ and a point x_0 where $w_\epsilon - W$ has a negative local minimum. So $w_\epsilon(x_0) < W(x_0) \leq G(x_0)$. Let $\Omega := \{w_\epsilon < W\}$ and observe that $w_\epsilon = W$ on $\partial\Omega$. Then x_0 is an interior point of Ω and

$$L(w_\epsilon) = -R^{-n} \ \text{ in } \ \Omega.$$

However

$$L(W - w_\epsilon) \geq -R^{-n} + R^{-n} = 0 \ \text{ in } \ \Omega. \tag{3.5}$$

By the weak maximum principle, the minimum can not be attained at an interior point, and so we have a contradiction. ∎

3.11 Lemma. *$w_0 = \tilde{w} = W$, and so w_0 and $\tilde{w}$ are continuous.*

Proof. We already showed that $w_0 = \tilde{w}$ in lemma (3.8). By lemma (3.10), in the set where $W = G$, we have

$$W = \tilde{w} = G. \tag{3.6}$$

Let $\Omega_1 := \{W < G\}$, it suffices to show $\tilde{w} = W$ in Ω_1. By definition of W, $L(W) = -R^{-n}$ in Ω_1.
Using the fact that w_0 is the minimizer, the standard argument in the calculus of variations leads to $L(w_0) \geq -R^{-n}$. Therefore

$$L(\tilde{w} - W) = L(w_0 - W) \geq 0 \ \text{ in } \ B_M. \tag{3.7}$$

Notice that on $\partial\Omega_1$, $W = \tilde{w} = G$. By weak maximum principle, we have

$$\tilde{w} = W \ \text{ in } \ \Omega_1. \tag{3.8}$$

∎

Using the last lemma along with our definition of W (see Equation (3.4)) we can now state the following theorem.

3.12 Theorem (The PDE satisfied by w_0)**.** *The minimizing function w_0 satisfies the following boundary value problem:*

$$\begin{cases} L(w_0) = -\chi_{\{w_0 < G\}} R^{-n} & \textit{in} \ \ B_M \\ \\ w_0 = G_{sm} & \textit{on} \ \ \partial B_M \ . \end{cases} \tag{3.9}$$

4 Optimal Regularity and Nondegeneracy

In order to proceed, we will need some basic results for the following obstacle type problem:

$$L(w) = \chi_{\{w>0\}} f \quad \text{in } B_1 \tag{4.1}$$

where we assume

$$w \geq 0 \quad \text{and} \quad 0 < \bar{\lambda} \leq f \leq \bar{\Lambda}\,. \tag{4.2}$$

All constants in this section will depend on $\lambda, \bar{\lambda}, \Lambda, \bar{\Lambda}$, and n, and f will always satisfy the inequalities above. We have two main results in this section: First, solutions to the problem above enjoy a parabolic bound from above at any free boundary point, and second, they have a quadratic nondegenerate growth from such points.

4.1 Lemma. *Let w satisfy Equations (4.1) and (4.2), and assume further that $w(0) = 0$. Then there exists a $\tilde{C}$ such that*

$$\| w \|_{L^\infty(B_{1/2})} \leq \tilde{C}. \tag{4.3}$$

Proof. Let u solve the following PDE:

$$\begin{cases} Lu = \chi_{\{w>0\}} f & \text{in } B_1 \\ \\ u = 0 & \text{on } \partial B_1\,. \end{cases} \tag{4.4}$$

Then Theorem 8.16 of [GT] gives

$$\| u \|_{L^\infty(B_1)} \leq C_1. \tag{4.5}$$

Now, consider the solution to:

$$\begin{cases} Lv = 0 & \text{in } B_1 \\ \\ v = w & \text{on } \partial B_1\,. \end{cases} \tag{4.6}$$

Notice that $u(x) + v(x) = w(x)$, and in particular $0 = w(0) = u(0) + v(0)$. Then by the Weak Maximum Principle and the Harnack Inequality, we have

$$\sup_{B_{1/2}} |v| = \sup_{B_{1/2}} v \leq C_2 \inf_{B_{1/2}} v \leq C_2 v(0) \leq C_2(-u(0)) \leq C_2 \cdot C_1. \tag{4.7}$$

Therefore

$$\| w \|_{L^\infty(B_{1/2})} \leq C \tag{4.8}$$

■

4.2 Theorem (Optimal Regularity)**.** *If $0 \in \partial\{w > 0\}$, then for any $x \in B_{1/2}$ we have*

$$w(x) \leq 4\tilde{C}|x|^2 \tag{4.9}$$

where $\tilde{C}$ is the same constant as in the statement of Lemma (4.1).

Proof. By the previous lemma, we know $\| w \|_{L^\infty(B_{1/2})} \leq \tilde{C}$. Notice that for any $\gamma > 1$,

$$u_\gamma(x) := \gamma^2 w\left(\frac{x}{\gamma}\right) \tag{4.10}$$

is also a solution to the same type of problem on B_1, but with a new operator $\tilde{L}$, and with a new function $\tilde{f}$ multiplying the characteristic function on the right hand side. On the other hand, the new operator has the same ellipticity as the old operator, and the new function $\tilde{f}$ has the same bounds that f had. Suppose there exist some point $x_1 \in B_{1/2}$ such that

$$w(x_1) > 4\tilde{C}|x_1|^2. \tag{4.11}$$

Then since $\frac{1}{2|x_1|} > 1$ and since $\frac{x_1}{2|x_1|} \in \partial B_{\frac{1}{2}}$, we have

$$u_{\left(\frac{1}{2|x_1|}\right)}\left(\frac{x_1}{2|x_1|}\right) = \frac{1}{4|x_1|^2} w(x_1) > \tilde{C} \ , \tag{4.12}$$

which contradicts Lemma (4.1). ∎

Now we turn to the nondegeneracy statement.

4.3 Lemma. *Let w satisfy the following*

$$L(w) \geq \bar{\lambda} \ \text{ in } B_r \quad \text{ and } w \geq 0 \ , \tag{4.13}$$

then there exists a positive constant, C, such that

$$\sup_{\partial B_r} w \geq w(0) + Cr^2 \ . \tag{4.14}$$

Proof. Let u solve

$$L(u) = 0 \ \text{ in } B_r \quad \text{ and } u = w \ \text{ on } \ \partial B_r \ . \tag{4.15}$$

Then the Weak Maximum Principle gives:

$$\sup_{\partial B_r} u \geq u(0). \tag{4.16}$$

Let v solve

$$L(v) = L(w) \geq \bar{\lambda} \ \text{ in } B_r \quad \text{ and } v = 0 \ \text{ on } \ \partial B_r \ . \tag{4.17}$$

Notice that $v_0(x) := \frac{|x|^2-r^2}{2n}$ solves

$$\Delta(v_0) = 1 \text{ in } B_r \quad \text{and } v_0 = 0 \text{ on } \partial B_r \, . \tag{4.18}$$

By [LSW], there exist constants C_1, C_2, such that $C_1 v_0(x) \leq v(x) \leq C_2 v_0(x)$ in $B_{r/2}$. In particular,

$$-v(0) \geq C_2 \frac{r^2}{2n}. \tag{4.19}$$

By the definitions of u and v, we know $w = u + v$, therefore by Equations (4.16) and (4.19) we have

$$\sup_{\partial B_r} w(x) = \sup_{\partial B_r} u(x) \geq u(0) = w(0) - v(0) \geq w(0) + C_2 \frac{r^2}{2n} \, . \tag{4.20}$$

∎

4.4 Lemma. *Let w solve the free boundary problem*

$$L(w) = \chi_{\{w>0\}} f \quad \text{in } B_1 \quad \text{and } w \geq 0 \, , \tag{4.21}$$

and $w(0) = \gamma > 0$, then $w > 0$ in a ball B_{δ_0} where $\delta_0 = C_0\sqrt{\gamma}$.

Proof. By Theorem (4.2), we know that if $w(x_0) = 0$, then

$$\gamma = |w(x_0) - w(0)| \leq C|x_0|^2, \tag{4.22}$$

which implies $|x_0| \geq C\sqrt{\gamma}$. ∎

4.5 Lemma (Nondegenerate Increase on a Polygonal Curve)**.** *Let w solve the free boundary problem*

$$L(w) = \chi_{\{w>0\}} f \quad \text{in } B_2 \quad \text{and } w \geq 0 \, , \tag{4.23}$$

and suppose $w(0) = \gamma > 0$. Then there exists a positive constant, C, such that

$$\sup_{B_1} w(x) \geq C + \gamma. \tag{4.24}$$

Proof. We can assume without loss of generality that there exists a $y \in B_{1/3}$ such that $w(y) = 0$. Otherwise we can apply the maximum principle along with Lemma (4.3) to get:

$$\sup_{B_1} w(x) \geq \sup_{B_{1/3}} w(x) \geq \gamma + C, \tag{4.25}$$

and we would already be done.

By Lemmas (4.3) and (4.4), there exist $x_1 \in \partial B_{\delta_0}$, such that

$$w(x_1) \geq w(0) + C\frac{\delta_0^2}{2n} = (1+C_1)\gamma \tag{4.26}$$

For this x_1 and $B_{\delta_1}(x_1)$ where $\delta_1 = C_0\sqrt{w(x_1)}$, Lemma (4.4) guarantees the existence of an $x_2 \in \partial B_{\delta_1}(x_1)$, such that

$$w(x_2) \geq (1+C_1)w(x_1) \geq (1+C_1)^2\gamma \tag{4.27}$$

Repeating the steps we can get finite sequences $\{x_i\}$ and $\{\delta_i\}$ with $x_0 = 0$ such that

$$w(x_i) \geq (1+C_1)^i\gamma \quad \text{and} \quad \delta_i = |x_{i+1} - x_i| = C_0\sqrt{w(x_i)}. \tag{4.28}$$

Observe that as long as $x_i \in B_{1/3}$, because of the existence of y where $w(y) = 0$ we know that $\delta_i \leq 2/3$, and so x_i is still in B_1. Pick N to be the smallest number which satisfies the following inequality:

$$\Sigma_{i=0}^N \delta_i = \Sigma_{i=0}^N C_0\sqrt{\gamma}(1+C_1)^{\frac{i}{2}} \geq \frac{1}{3}, \tag{4.29}$$

that is

$$N \geq \frac{2\ln\left[\frac{(1+C_1)^{\frac{1}{2}}-1}{3C_0\sqrt{\gamma}} + 1\right]}{\ln(1+C_1)} - 1. \tag{4.30}$$

Plugging this into Equation (4.28) gives

$$\begin{aligned}
w(x_N) &\geq \gamma(1+C_1)^{\frac{2\ln\left[\frac{(1+C_1)^{\frac{1}{2}}-1}{3C_0\sqrt{\gamma}}+1\right]}{\ln(1+C_1)}-1} \\
&= \frac{\gamma}{1+C_1}\left(\frac{(1+C_1)^{\frac{1}{2}}-1}{3C_0\sqrt{\gamma}} + 1\right)^2 \\
&= (\tilde{C}_0 + \tilde{C}_1\sqrt{\gamma})^2 \\
&\geq C_2(1+\gamma)\ .
\end{aligned}$$

■

4.6 Lemma. *Let w solve the free boundary problem*

$$L(w) = \chi_{\{w>0\}}f \quad \text{in } B_1 \quad \text{and } w \geq 0\ , \tag{4.31}$$

and $0 \in \overline{\{w>0\}}$. *Then*

$$\sup_{\partial B_1} w(x) \geq C. \tag{4.32}$$

Proof. By applying the maximum principle and the previous lemma this lemma is immediate. ∎

4.7 Theorem (Nondegeneracy)**.** *With $C = C(n, \lambda, \Lambda, \bar{\lambda}, \bar{\Lambda}) > 0$ exactly as in the previous lemma, and if $0 \in \overline{\{w > 0\}}$, then for any $r \leq 1$ we have*

$$\sup_{x \in B_r} w(x) \geq Cr^2 \,. \tag{4.33}$$

Proof. Assume there exists some $r_0 \leq 1$, such that

$$\sup_{x \in B_{r_0}} w(x) = C_1 r_0^2 < C{r_0}^2 \,. \tag{4.34}$$

Notice that for $\gamma \leq 1$,

$$u_\gamma(x) := \frac{w(\gamma x)}{\gamma^2} \tag{4.35}$$

is also a solution to the same type of problem with a new operator $\tilde{L}$ and new function $\tilde{f}$ defined in B_1, but the new operator has the same ellipticity as the old operator, and the new $\tilde{f}$ has the same bounds from above and below that f had. Now in particular for $u_{r_0}(x) = \frac{w(r_0 x)}{{r_0}^2}$, we have for any $x \in B_1$

$$u_{r_0}(x) = \frac{w(r_0 x)}{{r_0}^2} \leq \frac{1}{r_0^2} \sup_{x \in B_{r_0}} w(x) = C_1 < C \,, \tag{4.36}$$

which contradicts the previous lemma. ∎

5 Minimizers Become Independent of M

At this point we are no longer interested in the functions from the last section, with the exception of w_0. On the other hand, we now care about the dependence of w_0 on the radius of the ball on which it is a minimizer. Accordingly, we reintroduce the dependence of w_0 on M, and so we will let w_M be the minimizer of $J(w, B_M)$ within $K(M, G)$, and consider the behavior as $M \to \infty$. As we observed in Remark (3.2), it is not possible to start by minimizing our functional on all of $\mathbb{R}^n$, so we have to get the key function, "V_R," mentioned by Caffarelli on page 9 of [C] by taking a limit over increasing sets. Note that by Theorem (3.12) we know that w_M satisfies

$$\begin{cases} L(w_M) = -\chi_{\{G > w_M\}} R^{-n} & \text{in } \; B_M \\ \\ w_M = G_{sm} & \text{on } \; \partial B_M \,. \end{cases} \tag{5.1}$$

The theorem that we wish to prove in this section is the following:

5.1 Theorem (Independence from M)**.** *There exists $M \in \mathbb{N}$ such that if $M_j > M$ for $j = 1, 2$, then*

$$w_{M_1} \equiv w_{M_2} \quad \textit{within} \quad B_M$$

and

$$w_{M_1} \equiv w_{M_2} \equiv G \quad \textit{within} \quad B_{M+1} \setminus B_M \ .$$

Furthermore, we can choose M such that $M < C(n, \lambda, \Lambda) \cdot R$.

This Theorem is an immediate consequence of the following Theorem:

5.2 Theorem (Boundedness of the Noncontact Set)**.** *There exists a constant $C = C(n, \lambda, \Lambda)$ such that for any $M \in \mathbb{R}$*

$$\{w_M \neq G\} \subset B_{CR} \ . \tag{5.2}$$

Proof. First of all, if $M \leq CR$, then there is nothing to prove. For all $M > 1$ the function $W := G - w_M$ will satisfy:

$$L(W) = R^{-n}\chi_{\{W>0\}}, \text{ and } 0 \leq W \leq G \text{ in } B_1^c. \tag{5.3}$$

If the conclusion to the theorem is false, then there exists a large M and a large C such that

$$x_0 \in FB(W) \cap \{B_{M/2} \setminus B_{CR}\} \ .$$

Let $K := |x_0|/3$. By Theorem (4.7), we can then say that

$$\sup_{B_K(x_0)} W(x) \geq CR^{-n}K^2 > CK^{2-n} \geq \sup_{B_K(x_0)} G(x) \tag{5.4}$$

which gives us a contradiction since $W \leq G$ everywhere. Now note that in order to avoid the contradiction, we must have

$$CR^{-n}K^2 \leq CK^{2-n} \ ,$$

and this leads to

$$K \leq CR$$

which means that $|x_0|$ must be less than CR. In other words, $FB(W) \subset B_{CR}$. ∎

At this point, we already know that when M is sufficiently large, the set $\{G > w_M\}$ is contained in B_{CR}. Then by uniqueness, the set will stay the same for any bigger M. Therefore, it makes sense to define w_R to be the solution of

$$Lw = -R^{-n}\chi_{\{w<G\}} \quad \text{in } \mathbb{R}^n \tag{5.5}$$

among functions $w \leq G$ with $w = G$ at infinity. Note that we can now obtain the function, "V_R," that Caffarelli uses on page 9 of [C]. The relationship is simply:

$$V_R = w_R - G \ . \tag{5.6}$$

6 The Mean Value Theorem

Finally, we can turn to the Mean Value Theorem.

6.1 Lemma (Ordering of Sets)**.** *For any $R < S$, we have*

$$\{w_R < G\} \subset \{w_S < G\}. \tag{6.1}$$

Proof. Let B_M be a ball that contains both $\{w_R < G\}$ and $\{w_S < G\}$. Then by the discussion in Section 2, we know w_R minimizes

$$\int_{B_M} a^{ij} D_i w D_j w - 2wR^{-n}$$

and w_S minimizes

$$\int_{B_M} a^{ij} D_i w D_j w - 2wS^{-n}.$$

Let $\Omega_1 \subset\subset B_M$ be the set $\{w_S > w_R\}$. Then it follows that

$$\int_{\Omega_1} a^{ij} D_i w_S D_j w_S - 2w_S S^{-n} \le \int_{\Omega_1} a^{ij} D_i w_R D_j w_R - 2w_R S^{-n}, \tag{6.2}$$

which implies

$$\begin{aligned} \int_{\Omega_1} a^{ij} D_i w_S D_j w_S &\le \int_{\Omega_1} a^{ij} D_i w_R D_j w_R + 2S^{-n} \int_{\Omega_1} (w_S - w_R) \\ &< \int_{\Omega_1} a^{ij} D_i w_R D_j w_R + 2R^{-n} \int_{\Omega_1} (w_S - w_R). \end{aligned}$$

Therefore, since $w_S \equiv w_R$ on $\partial\Omega_1$, and

$$\int_{\Omega_1} a^{ij} D_i w_S D_j w_S - 2w_S R^{-n} < \int_{\Omega_1} a^{ij} D_i w_R D_j w_R - 2w_R R^{-n}, \tag{6.3}$$

we contradict the fact that w_R is the minimizer of $\int a^{ij} D_i w D_j w - 2wR^{-n}$. ∎

6.2 Lemma. *There exists a constant $c = c(n, \lambda, \Lambda)$ such that*

$$B_{cR} \subset \{G > w_R\}.$$

Proof. By Lemma (3.4) we already know that there exists a constant

$$C = C(n, \lambda, \Lambda)$$

such that $w_1(0) \le C$. Then it is not hard to show that

$$\|w_1\|_{L^\infty(B_{1/2})} \le \tilde{C}. \tag{6.4}$$

By [LSW] for any elliptic operator L with given λ and Λ, we have

$$\frac{c_1}{|x|^{n-2}} \leq G(x) \leq \frac{c_2}{|x|^{n-2}}. \tag{6.5}$$

By combining the last two equations it follows that there exists a constant $c = c(n, \lambda, \Lambda)$ such that

$$B_c \subset \{G > w_1\}.$$

It remains to show that this inclusion scales correctly.

Let $v_R := G - w_R$ (so $v_R = -V_R$). Then v_R satisfies

$$Lv_R = \delta - R^{-n}\chi_{\{v_R>0\}} \quad \text{in } \mathbb{R}^n \,. \tag{6.6}$$

Now observe that by scaling our operator L appropriately, we get an operator $\tilde{L}$ with the same ellipticity constants as L, such that

$$\tilde{L}\left(R^{n-2}v_R(Rx)\right) = \delta - \chi_{\{v_R(Rx)>0\}} \,. \tag{6.7}$$

So we have

$$B_c \subset \left\{x \,\middle|\, v_R(Rx) > 0\right\},$$

which implies

$$B_{cR} \subset \left\{v_R(x) > 0\right\}. \tag{6.8}$$

■

Suppose v is a supersolution to

$$Lv = 0,$$

i.e. $Lv \leq 0$. Then for any $\phi \geq 0$, we have

$$\int_\Omega vL\phi \leq 0. \tag{6.9}$$

If $R < S$, then we know that $w_R \geq w_S$, and so the function $\phi = w_R - w_S$ is a permissible test function. We also know:

$$L\phi = R^{-n}\chi_{\{G>w_R\}} - S^{-n}\chi_{\{G>w_S\}}. \tag{6.10}$$

By observing that $v \equiv 1$ is both a supersolution and a subsolution and by plugging in our ϕ, we arrive at

$$R^{-n}|\{G > w_R\}| = S^{-n}|\{G > w_S\}|, \tag{6.11}$$

and this implies

$$L\phi = C\left[\frac{1}{|\{G > w_R\}|}\chi_{\{G>w_R\}} - \frac{1}{|\{G > w_S\}|}\chi_{\{G>w_S\}}\right]. \tag{6.12}$$

Now, Equation (6.9) implies

$$0 \geq \int_\Omega vL\phi = C\left[\frac{1}{|\{G > w_R\}|}\int_{\{G>w_R\}} v - \frac{1}{|\{G > w_S\}|}\int_{\{G>w_S\}} v\right]. \tag{6.13}$$

Therefore, we have established the following theorem:

6.3 Theorem (Mean Value Theorem for Divergence Form Elliptic PDE)**.** *Let L be any divergence form elliptic operator with ellipticity λ, Λ. For any $x_0 \in \Omega$, there exists an increasing family $D_R(x_0)$ which satisfies the following:*

1. *$B_{cR}(x_0) \subset D_R(x_0) \subset B_{CR}(x_0)$, with c, C depending only on n, λ and Λ.*
2. *For any v satisfying $Lv \geq 0$ and $R < S$, we have*

$$v(x_0) \leq \frac{1}{|D_R(x_0)|}\int_{|D_R(x_0)|} v \leq \frac{1}{|D_S(x_0)|}\int_{D_S(x_0)} v. \tag{6.14}$$

As on pages 9 and 10 of [C], (and as Littman, Stampacchia, and Weinberger already observed using their own mean value theorem,) we have the following corollary:

6.4 Corollary (Semicontinuous Representative)**.** *Any supersolution v, has a unique pointwise defined representative as*

$$v(x_0) := \lim_{R\downarrow 0}\frac{1}{|D_R(x_0)|}\int_{|D_R(x_0)|} v(x)dx\ . \tag{6.15}$$

This representative is lower semicontinuous:

$$v(x_0) \leq \lim_{x\to x_0} v(x) \tag{6.16}$$

for any x_0 in the domain.

We can also show the following analogue of G.C. Evans' Theorem:

6.5 Corollary (Analogue of Evans' Theorem)**.** *Let v be a supersolution to $Lv = 0$, and suppose that v restricted to the support of Lv is continuous. Then the representative of v given by Equation (6.16) is continuous.*

Proof. This proof is almost identical to the proof given on pages 10 and 11 of [C] for $L = \Delta$. ∎

7 Acknowledgements

We are endebted to both Luis Caffarelli and Luis Silvestri for very useful conversations.

References

[C] L.A. Caffarelli, The Obstacle Problem. The Fermi Lectures, Accademia Nazionale Dei Lincei Scuola Normale Superiore, 1998.

[GT] D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd ed., Springer-Verlag, 1983.

[G] E. Giusti, Direct Methods in the Calculus of Variations, World Scientific, 2003.

[HL] Q. Han and F. Lin, Elliptic Partial Differential Equations, AMS, 2000.

[LSW] W. Littman, G. Stampacchia, and H.F. Weinberger, Regular points for elliptic equations with discontinuous coefficients, *Ann. Scuola. Norm. Sup. Pisa. Cl. Sci.,* 17(1963), no. 1-2, 43–77.